Solutions to discrete Painlevé systems arising from two types of orthogonal polynomials (I)

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Abstract. We consider the relation between discrete Painlevé systems and orthogonal polynomials associated with the Christoffel transformation. We construct a method to obtain the particular solutions to discrete Painlevé systems by using orthogonal polynomials and their kernel polynomials. In particular, we treat the cases of the Hermite polynomials and the discrete q-Hermite II polynomials as examples.

Keywords and Phrases: discrete Painlevé equation; orthogonal polynomial; Christoffel transformation; discrete Riccati equation; partition function
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1 Introduction

In the 1900s, the Painlevé equations, $\mathrm{P_I}$, $\mathrm{P_{II}}$, ... and $\mathrm{P_{VI}}$, were defined by P. Painlevé, R. Fuchs and B. Gambier. The Painlevé equations are nonlinear ordinary differential equations of second order that posses no movable singular point. There are many works that investigate the properties of the Painlevé equations [29, 30, 31, 32, 33]D Discrete Painlevé systems are nonlinear ordinary difference equations of second order and known as discrete versions of Painlevé equations. It is known that there are three difference types (additive type, multiplicative type and elliptic type) for discrete Painlevé systems. There are also many works that investigate the properties of discrete Painlevé systems [8, 16, 17, 37]D In [37], H. Sakai introduced a geometric approach to the theory of Painlevé systems and showed the classifications of Painlevé equations and discrete Painlevé systems by the rational surfacesD The rational surface can be identified with the space of initial condition, and the group of Cremona isometries associated with the surface generate the affine Weyl group.

Some discrete Painlevé systems have been found in the studies of random matrices [4, 12, 34]D As one such example, let us consider the partition function of the Gaussian Unitary Ensemble of an $n \times n$ random matrix:

$$Z_n^{(2)} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Delta(t_1, \cdots, t_n)^2 \prod_{i=1}^{n} e^{\eta(t_i)} \mathrm{d}t_i, \quad \eta(t_i) = \sum_{m=0}^{\infty} z_m {t_i}^m, \tag{1.1}$$

where $\Delta(t_1, \cdots, t_n)$ is Vandermonde's determinant. Note that throughout this paper we assume

$$\prod_{i=1}^{n} f(i) = 1, \quad \prod_{i=0}^{n-1} f(i) = 1, \quad \Delta(t_1, \cdots, t_n) = 1, \tag{1.2}$$

for an arbitrary function $f(i)$ when $n = 0$. Here we choose

$$\eta(t_i) = -g_1 {t_i}^2 - g_2 {t_i}^4, \quad (g_2 > 0). \tag{1.3}$$

Setting

$$R_n = \frac{Z_{n+1}^{(2)} Z_{n-1}^{(2)}}{\left(Z_n^{(2)}\right)^2}, \tag{1.4}$$

we obtain the following difference equation[4, 10, 11, 36]:

$$R_{n+1}+R_n+R_{n-1}=\frac{n}{4g_2}\,\frac{1}{R_n}-\frac{g_1}{2g_2}\,. \tag{1.5}$$

Equation (1.5) is referred to as a discrete Painlevé I equation, denoted by d-$\mathrm{P_I}$, and has the space of initial condition of type $E_6^{(1)}$. Such relations between discrete Painlevé systems and random matrices are well known.

Now, we introduce a q-version of a partition function, using (1.1) as our reference. We consider $\psi_n^{l,m}$ $(l,n\in\mathbb{Z}_{\geq 0},\ m\in\mathbb{Z},\ a\in\mathbb{C},\ c_1\in\mathbb{R}_{>0})$ given as

$$\psi_n^{l,m}=\frac{q^{n(n-1)(2l-1)/2}}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Delta(t_1,\cdots,t_n)^2\prod_{i=1}^{n}\frac{\prod_{j=0}^{l-1}(q^jt_i-q^ma)}{E_{q^2}\left({c_1}^2{t_i}^2\right)}\,\mathrm{d}_qt_i. \tag{1.6}$$

The definitions of the q-definite integral $\int_{-\infty}^{\infty}\mathrm{d}_qt$ and the q-exponential function $E_q(t)$ appearing here are given at the end of this section. As in the case of (1.1), we can obtain a solution to a discrete Painlevé equation expressible in terms of $\psi_n^{l,m}$. Specifically, we have the following:

Lemma 1.1. *A q-analogue of the Painlevé IV equation corresponding to the surface of type $A_4^{(1)}$ (q-$\mathrm{P_{IV}}$) [35, 39] :*

$$\begin{aligned}&(X_{n+1}X_n-1)(X_{n-1}X_n-1)\\&\quad=q^{-N+2n-m-1}a_0{a_1}^{3/2}{a_2}^2\frac{(X_n+q^{N-m}{a_1}^{1/2})(X_n+q^{-N+m}{a_1}^{-1/2})}{X_n+q^{-N+n-m}{a_1}^{1/2}a_2}\,,\end{aligned} \tag{1.7}$$

has the following solution:

$$X_n=i\,\frac{(1-q^{n+1})q^n}{c_1}\,\frac{\psi_{n+1}^{0,0}\psi_n^{1,-m}}{\psi_n^{0,0}\psi_{n+1}^{1,-m}}\,. \tag{1.8}$$

Here

$${a_0}^{1/2}=-iqa^{-1}{c_1}^{-1},\quad {a_0}^{1/2}{a_1}^{1/2}=q^{-N},\quad a_2=q^{2N+2},\quad a\neq 0. \tag{1.9}$$

The proof of Lemma 1.1 will be given in Appendix.

Below, we investigate the solutions to d-$\mathrm{P_I}$ and q-$\mathrm{P_{IV}}$ from the viewpoint of orthogonal polynomials. First, however, we define orthogonal polynomials:

Definition 1.1. A polynomial sequence $(P_n(t))_{n=0}^{\infty}$ which satisfies the following conditions is called an orthogonal polynomial sequence over the field $\mathcal{K}$, and each term $P_n(t)$ is called an orthogonal polynomial over the field $\mathcal{K}$.

- $\deg(P_n(t))=n\mathrm{D}$
- There exists a linear functional $\mathcal{L}:\mathcal{K}(t)\to\mathcal{K}$ which holds the orthogonal condition:

$$\mathcal{L}[t^kP_n(t)]=h_n\delta_{n,k},\quad (n\geq k), \tag{1.10}$$

where $\delta_{n,k}$ is Kronecker's symbol. Here, h_n is called the normalization factor and $\mu_n=\mathcal{L}[t^n]$ $(n=0,1,\dots)$ is called the moment sequence.

Definition 1.2. An orthogonal polynomial sequence whose coefficient of leading term is 1 is called a monic orthogonal polynomial sequence (MOPS). Let $(P_n(t))_{n=0}^{\infty}$ be MOPS. $P_n(t)$ and its normalization factor h_n are given as

$$P_n(t) = \frac{1}{\tau_n} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & t & \cdots & t^n \end{vmatrix}, \quad h_n = \frac{\tau_{n+1}}{\tau_n}. \tag{1.11}$$

Here $(\mu_n)_{n=0}^{\infty}$ is the moment sequence and τ_n is the Hankel determinant given as

$$\tau_0 = 1, \quad \tau_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} \quad (n \in \mathbb{Z}_{>0}). \tag{1.12}$$

First, we reconsider the solution to d-$\mathrm{P_I}$ appearing in (1.4). Let $(P_n(t))_{n=0}^{\infty}$ be MOPS defined as

$$\mathcal{L}[P_n(t)P_k(t)] = \int_{-\infty}^{\infty} P_n(t)P_k(t)e^{-g_1t^2-g_2t^4}\mathrm{d}t = h_n\delta_{n,k}. \tag{1.13}$$

The moment is given as

$$\mu_n = \int_{-\infty}^{\infty} t^n e^{-g_1t^2-g_2t^4}\mathrm{d}t = \begin{cases} \dfrac{(2k)!\sqrt{\pi}e^{{g_1}^2/8g_2}}{2^{2k+1}(2g_2)^{(2k+1)/4}k!}\, D_{-k-\frac{1}{2}}\left(\dfrac{g_1}{\sqrt{2g_2}}\right) & (n = 2k), \\ 0 & (n = 2k+1), \end{cases} \tag{1.14}$$

and the normalization factor is

$$h_n = \frac{Z_{n+1}^{(2)}}{Z_n^{(2)}}. \tag{1.15}$$

Here $D_\lambda(z)$ is the parabolic cylinder function defined as

$$D_\lambda(z) = \frac{e^{-z^2/4}}{\Gamma(-\lambda)} \int_0^{\infty} e^{-t^2/2-zt}t^{-(\lambda+1)}\mathrm{d}t, \quad (\mathrm{Re}(\lambda) < 0), \tag{1.16}$$

which satisfies

$$\frac{\mathrm{d}^2 D_\lambda(z)}{\mathrm{d}z^2} + \left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right) D_\lambda(z) = 0. \tag{1.17}$$

P_n satisfies the following three-term recurrence relation:

$$tP_n(t) = P_{n+1}(t) + \frac{h_n}{h_{n-1}}\, P_{n-1}. \tag{1.18}$$

Substituting $n = k$ in (1.13) and then applying partial integration on it, we obtain

$$(1+2n)h_n = \int_{-\infty}^{\infty} tP_n(t)^2(2g_1t + 4g_2t^3)e^{-g_1t^2-g_2t^4}\mathrm{d}t. \tag{1.19}$$

From compatibility conditions of (1.18) and (1.19), we find that

$$R_n = \frac{h_n}{h_{n-1}}\,, \tag{1.20}$$

is the solution to d-P$_{\rm I}$. Instead of (1.19) we can use a differential equation or a difference equation for $P_n(t)$. In any case, $R_n = h_n/h_{n-1}$ essentially becomes a variable of the discrete Painlevé equation.

We next reconsider the solution to q-P$_{\rm IV}$ appearing in (1.8). The Hankel determinant expression of $\psi_n^{l,m}$ is given by the following lemma:

Lemma 1.2. *$\psi_n^{l,m}$, given in* (1.6)*, can be expressed as*

$$\psi_n^{l,m} = q^{n(n-1)(2l-1)/2}\begin{vmatrix} H_{l,m,0} & H_{l,m,1} & \cdots & H_{l,m,n-1}\\ H_{l,m,1} & H_{l,m,2} & \cdots & H_{l,m,n}\\ \vdots & \vdots & \ddots & \vdots\\ H_{l,m,n-1} & H_{l,m,n} & \cdots & H_{l,m,2n-2}\end{vmatrix}. \tag{1.21}$$

Here, the entries are given as

$$H_{l,m,k} = \int_{-\infty}^{\infty} t^k\,\frac{\prod_{j=0}^{l-1}(q^j t - q^m a)}{E_{q^2}\left({c_1}^2 t^2\right)}\,{\rm d}_q t. \tag{1.22}$$

Proof. Equation (1.21) can be verified by the following calculation:

$$\begin{aligned}\psi_n^{l,m} =&\frac{q^{n(n-1)(2l-1)/2}}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Delta(t_1,\cdots,t_n)\sum_{\sigma\in\mathfrak{S}_{\mathbf{n}}}({\rm sgn}\ \sigma){t_{\sigma(1)}}^0{t_{\sigma(2)}}^1\cdots{t_{\sigma(n)}}^{n-1}\\ &\times\prod_{i=1}^n\frac{\prod_{j=0}^{l-1}(q^j t_i - q^m a)}{E_{q^2}\left({c_1}^2 {t_i}^2\right)}\,{\rm d}_q t_i \qquad (1.23)\\ =&\frac{q^{n(n-1)(2l-1)/2}}{n!}\sum_{\sigma\in\mathfrak{S}_{\mathbf{n}}}({\rm sgn}\ \sigma)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\begin{vmatrix} 1 & 1 & \cdots & 1\\ t_1 & t_2 & \cdots & t_n\\ \vdots & \vdots & \ddots & \vdots\\ {t_1}^{n-1} & {t_2}^{n-1} & \cdots & {t_n}^{n-1}\end{vmatrix}\\ &\times {t_1}^{\sigma(1)-1}{t_2}^{\sigma(2)-1}\cdots{t_n}^{\sigma(n)-1}\prod_{i=1}^n\frac{\prod_{j=0}^{l-1}(q^j t_i - q^m a)}{E_{q^2}\left({c_1}^2 {t_i}^2\right)}\,{\rm d}_q t_i \qquad (1.24)\\ =&q^{n(n-1)(2l-1)/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\begin{vmatrix} 1 & t_2 & \cdots & {t_n}^{n-1}\\ t_1 & {t_2}^2 & \cdots & {t_n}^n\\ \vdots & \vdots & \ddots & \vdots\\ {t_1}^{n-1} & {t_2}^{n} & \cdots & {t_n}^{2n-2}\end{vmatrix}\prod_{i=1}^n\frac{\prod_{j=0}^{l-1}(q^j t_i - q^m a)}{E_{q^2}\left({c_1}^2 {t_i}^2\right)}\,{\rm d}_q t_i\\ &\qquad (1.25)\\ =&q^{n(n-1)(2l-1)/2}\begin{vmatrix} H_{l,m,0} & H_{l,m,1} & \cdots & H_{l,m,n-1}\\ H_{l,m,1} & H_{l,m,2} & \cdots & H_{l,m,n}\\ \vdots & \vdots & \ddots & \vdots\\ H_{l,m,n-1} & H_{l,m,n} & \cdots & H_{l,m,2n-2}\end{vmatrix}. \qquad (1.26)\end{aligned}$$

■

Letting $(P_n^{l,m})_{n=0}^{\infty}$ be MOPS defined as

$$\int_{-\infty}^{\infty} P_n^{l,m}(t) P_k^{l,m}(t)\ \frac{\prod_{j=0}^{l-1}(q^j t - q^m a)}{E_{q^2}\left({c_1}^2 t^2\right)}\ \mathrm{d}_q t = h_n^{l,m}\delta_{n,k}, \tag{1.27}$$

we can regard $(H_{l,m,n})_{n=0}^{\infty}$ as a moment sequence. Therefore, by using the normalization factor

$$h_n^{l,m} = q^{(1-2l)n}\frac{\psi_{n+1}^{l,m}}{\psi_n^{l,m}}\,, \tag{1.28}$$

the solution to q-$\mathrm{P_{IV}}$ can be rewritten as

$$X_n = i\ \frac{1-q^{n+1}}{c_1 q^n}\ \frac{h_n^{0,0}}{h_n^{1,-m}}\,. \tag{1.29}$$

Note that $P_n^{0,m}(t)$ defined as (1.27) is referred to as the discrete q-Hermite II polynomial (cf. [19]) and $P_n^{1,m}(t)$ is said to the kernel polynomial of $P_n^{0,m}(t)$ given by the Christoffel transformation. The definitions of the kernel polynomial and the Christoffel transformation will be given in the next section.

The solution to d-$\mathrm{P_I}$, (1.20), is given by the single orthogonal polynomial, while that to q-$\mathrm{P_{IV}}$, (1.29), is expressed by the two different orthogonal polynomials. From this viewpoint, the types of solutions to d-$\mathrm{P_I}$ and q-$\mathrm{P_{IV}}$ are different. In the past the solutions to discrete Painlevé systems expressed in terms of normalization factor of one type of orthogonal polynomial has been studied[1, 2, 3, 4, 12, 34, 38], but as far as I know, there is no study about one expressed in terms of normalization factors of two types of orthogonal polynomial. The purpose of this paper is to construct the method to give the solutions expressed in terms of normalization factors of two types of orthogonal polynomials. We note here that solutions to the Painlevé equations expressed in terms of normalization factors of two types of orthogonal polynomial is studied in [5, 6].

This paper is organized as follows. In Section 2, we consider the compatibility conditions of an orthogonal polynomial and its kernel polynomial. In Section 3, we demonstrate with examples that from the compatibility condition given in Section 2 we can obtain the solution to the discrete Painlevé systems. Concluding remarks are given in Section 4.

Throughout this paper, we assume $0 < |q| < 1$ and the expression "α is a constant" means $\mathrm{d}\alpha/\mathrm{d}t = 0$, where t is the independent variable of the orthogonal polynomial. We use the following conventions of q-analysis[7, 19]D
q-Shifted factorials:

$$(a;q)_\infty = \prod_{i=1}^{\infty}(1-aq^{i-1}), \tag{1.30}$$

$$(a;q)_\lambda = \frac{(a;q)_\infty}{(aq^\lambda;q)_\infty}\,, \quad (\lambda\in\mathbb{C}). \tag{1.31}$$

Jacobi theta function:

$$\Theta(a;q) = (a;q)_\infty(qa^{-1};q)_\infty. \tag{1.32}$$

q-Exponential function:

$$E_q(z) = \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}z^n = (-z;q)_\infty. \tag{1.33}$$

q-Derivative:

$$D_{q,t}f(t)=\frac{f(qt)-f(t)}{(q-1)t}\,. \tag{1.34}$$

q-Definite integral:

$$\int_{-\infty}^{\infty} f(t)\mathrm{d}_qt=(1-q)\sum_{n=-\infty}^{\infty}(f(q^n)+f(-q^n))q^n. \tag{1.35}$$

Basic hypergeometric series:

$${}_s\varphi_r\left(\begin{matrix}a_1,\cdots,a_s\\ b_1,\cdots,b_r\end{matrix};q,z\right)=\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_s;q)_n}{(b_1,\cdots,b_r;q)_n(q;q)_n}\left[(-1)^nq^{n(n-1)/2}\right]^{1+r-s}z^n, \tag{1.36}$$

where

$$(a_1,\cdots,a_s;q)_n=\prod_{j=1}^{s}(a_j;q)_n. \tag{1.37}$$

Bilateral basic hypergeometric series:

$${}_s\psi_r\left(\begin{matrix}a_1,\cdots,a_s\\ b_1,\cdots,b_r\end{matrix};q,z\right)=\sum_{n=-\infty}^{\infty}\frac{(a_1,\cdots,a_s;q)_n}{(b_1,\cdots,b_r;q)_n}\left[(-1)^nq^{n(n-1)/2}\right]^{r-s}z^n. \tag{1.38}$$

Finally, we note that the following relations hold:

$$(a;q)_\lambda=\frac{(a;q)_{\lambda+1}}{1-aq^\lambda}\,, \tag{1.39}$$

$$\Theta(a;q)=-a\Theta(qa;q), \tag{1.40}$$

$$\int_{-\infty}^{\infty} f(t)\mathrm{d}_qt=q\int_{-\infty}^{\infty} f(qt)\mathrm{d}_qt, \tag{1.41}$$

$$\begin{aligned}&\int_{-\infty}^{\infty} f(t)\big(D_{q,t}g(t)\big)\mathrm{d}_qt\\ &\quad=\lim_{n\to\infty}\big(f(q^{-n})g(q^{-n})-f(-q^{-n})g(-q^{-n})\big)-\int_{-\infty}^{\infty}\big(D_{q,t}f(t)\big)g(qt)\mathrm{d}_qt.\end{aligned} \tag{1.42}$$

2 The compatibility conditions associated with the Christoffel transformation

In this section, we consider the compatibility conditions of an orthogonal polynomial and its kernel polynomial.

Let $(P_n)_{n=0}^{\infty}=(P_n(t))_{n=0}^{\infty}$ and $(\hat{P}_n)_{n=0}^{\infty}=(\hat{P}_n(t))_{n=0}^{\infty}$ be MOPSs with linear functionals $\mathcal{L}$ and $\hat{\mathcal{L}}$ over $\mathbb{C}$ given as

$$\mathcal{L}[t^kP_n(t)]=h_n\delta_{n,k} \qquad (n\geq k), \tag{2.1}$$

$$\hat{\mathcal{L}}[t^k\hat{P}_n(t)]=\mathcal{L}[(t-c_0)t^k\hat{P}_n(t)]=\hat{h}_n\delta_{n,k} \qquad (n\geq k,\ c_0\in\mathbb{C}), \tag{2.2}$$

respectively. μ_n and $\hat{\mu}_n$ are the moments given as

$$\mu_n=\mathcal{L}[t^n], \tag{2.3}$$

$$\hat{\mu}_n=\hat{\mathcal{L}}[t^n], \tag{2.4}$$

respectively. We refer to the transformation from P_n to $\hat{P}_n$ as the Christoffel transformation and $\hat{P}_n$ as the kernel polynomial. On the other hand, we also refer to the transformation from $\hat{P}_n$ to P_n as the Geronimus transformation. First we consider the relations between P_n and $\hat{P}_n$.

Lemma 2.1. *The following relations hold*:

$$(t-c_0)\hat{P}_n = P_{n+1} + \frac{\hat{h}_n}{h_n}\, P_n, \tag{2.5}$$

$$P_n = \hat{P}_n + \frac{h_n}{\hat{h}_{n-1}}\, \hat{P}_{n-1}. \tag{2.6}$$

Proof. We first prove (2.5). The polynomial $(t-c_0)\hat{P}_n$ can be expressed with certain constants C_j as

$$(t-c_0)\hat{P}_n = P_{n+1} + \sum_{j=0}^{n} C_j P_j, \tag{2.7}$$

and then it holds

$$\hat{\mathcal{L}}[\hat{P}_n P_k] = \mathcal{L}[P_{n+1}P_k] + \sum_{j=0}^{n} C_j \mathcal{L}[P_j P_k], \tag{2.8}$$

thus,

$$\hat{h}_n \delta_{n,k} = C_k h_k, \tag{2.9}$$

where $0 \le k \le n$. Therefore (2.5) holds.

We next prove (2.6). Setting

$$P_n = \hat{P}_n + \sum_{j=0}^{n-1} \hat{C}_j \hat{P}_j, \tag{2.10}$$

where $\hat{C}_j$ is a constant, we obtain

$$\mathcal{L}[tP_nP_k] - c_0\mathcal{L}[P_nP_k] = \hat{\mathcal{L}}[\hat{P}_nP_k] + \sum_{j=0}^{n-1} \hat{C}_j \hat{\mathcal{L}}[\hat{P}_jP_k], \tag{2.11}$$

thus,

$$h_n\delta_{n,k+1} = \sum_{j=0}^{k} \hat{C}_j \hat{\mathcal{L}}[\hat{P}_jP_k], \tag{2.12}$$

where $0 \le k \le n-1$. Therefore (2.6) holds. ■

We obtain the following lemma from the compatibility conditions of (2.5) and (2.6):

Lemma 2.2. *The following three-term recurrence relations hold*:

$$tP_n = P_{n+1} + \left(\frac{\hat{h}_n}{h_n} + \frac{h_n}{\hat{h}_{n-1}} + c_0\right) P_n + \frac{h_n}{h_{n-1}}\, P_{n-1}, \tag{2.13}$$

$$t\hat{P}_n = \hat{P}_{n+1} + \left(\frac{h_{n+1}}{\hat{h}_n} + \frac{\hat{h}_n}{h_n} + c_0\right) \hat{P}_n + \frac{\hat{h}_n}{\hat{h}_{n-1}}\, \hat{P}_{n-1}. \tag{2.14}$$

Proof. Eliminating $\hat{P}_n$ from (2.6) by using (2.5), we obtain (2.13), and eliminating P_n from (2.5) by using (2.6), we obtain (2.14). ■

We define the constants α_n, β_n, $\hat{\alpha}_n$ and $\hat{\beta}_n$ as

$$tP_n = P_{n+1} + \alpha_n P_n + \beta_n P_{n-1}, \tag{2.15}$$

$$t\hat{P}_n = \hat{P}_{n+1} + \hat{\alpha}_n \hat{P}_n + \hat{\beta}_n \hat{P}_{n-1}. \tag{2.16}$$

From (2.13) and (2.14), we obtain the following:

$$\alpha_n = \frac{\hat{h}_n}{h_n} + \frac{h_n}{\hat{h}_{n-1}} + c_0, \quad \beta_n = \frac{h_n}{h_{n-1}}, \tag{2.17}$$

$$\hat{\alpha}_n = \frac{h_{n+1}}{\hat{h}_n} + \frac{\hat{h}_n}{h_n} + c_0, \quad \hat{\beta}_n = \frac{\hat{h}_n}{\hat{h}_{n-1}}. \tag{2.18}$$

Set

$$x_n = \frac{h_n}{\hat{h}_n}, \quad y_n = \frac{\hat{h}_n}{h_{n+1}}. \tag{2.19}$$

From (2.17) and (2.18), we obtain

$$x_n = -\frac{1}{\beta_n x_{n-1} - \alpha_n + c_0}, \tag{2.20}$$

$$y_n = -\frac{1}{\hat{\beta}_n y_{n-1} - \hat{\alpha}_n + c_0}. \tag{2.21}$$

When we give an orthogonal polynomial P_n such that both α_n and β_n are rational functions of n (or q^n), we can regard (2.20) as a discrete Riccati equation. Similarly, when we give an orthogonal polynomial $\hat{P}_n$, (2.21) can be also regarded as a discrete Riccati equation. Therefore we find that the compatibility conditions of an orthogonal polynomial and its kernel polynomial can be related to the discrete Painlevé equation through the discrete Riccati equation. In the next section, we demonstrate this point with examples in the case where both α_n and β_n (or, $\hat{\alpha}_n$ and $\hat{\beta}_n$) are rational functions of n and in the case where both α_n and β_n (or, $\hat{\alpha}_n$ and $\hat{\beta}_n$) are rational functions of q^n.

3 Relation between the compatibility conditions and discrete Painlevé systems

In this section, we show that from (2.20) and (2.21) we can obtain the solutions to discrete Painlevé systems. We demonstrate the construction by taking two examples. The first example is the Hermite polynomials in the case where both α_n and β_n (or, $\hat{\alpha}_n$ and $\hat{\beta}_n$) are rational functions of n. The second one is the discrete q-Hermite II polynomials in the case where both α_n and β_n (or, $\hat{\alpha}_n$ and $\hat{\beta}_n$) are rational functions of q^n.

3.1 Example I: The case where $(P_n)_{n=0}^{\infty}$ are the Hermite polynomials

We define P_n as

$$P_n(t) = \frac{H_n(c_2 t + c_1)}{{c_2}^n}, \quad (c_2 > 0,\ c_1 \in \mathbb{C}), \tag{3.1}$$

where H_n is the Hermite polynomial:

$$H_n(t) = (-1)^n e^{t^2/2} \frac{\mathrm{d}^n}{\mathrm{d}t^n} (e^{-t^2/2}). \tag{3.2}$$

The linear functionals and the moments are given as

$$\mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-c_2{}^2 t^2/2 - c_1 c_2 t} \mathrm{d}t, \tag{3.3}$$

$$\hat{\mathcal{L}}[f(t)] = \int_{-\infty}^{\infty} (t - c_0) f(t) e^{-c_2{}^2 t^2/2 - c_1 c_2 t} \mathrm{d}t, \tag{3.4}$$

$$\mu_n = \frac{2^{1/2}\sqrt{\pi}}{c_2{}^{n+1}} e^{n\pi i/2 + c_1{}^2/4} D_n(ic_1), \tag{3.5}$$

$$\hat{\mu}_n = \mu_{n+1} - c_0 \mu_n, \tag{3.6}$$

where $D_\lambda(z)$ is the parabolic cylinder function defined in (1.16). From the three-term recurrence relation:

$$tP_n = P_{n+1} - \frac{c_1}{c_2} P_n + \frac{n}{c_2{}^2} P_{n-1}, \tag{3.7}$$

we obtain

$$\alpha_n = -\frac{c_1}{c_2}, \quad \beta_n = \frac{n}{c_2{}^2}. \tag{3.8}$$

From (2.20), we obtain the following discrete Riccati equation:

$$x_n = -\frac{c_2{}^2}{n x_{n-1} + c_2(c_1 + c_0 c_2)}. \tag{3.9}$$

We consider the following difference equation[9, 10, 11, 26, 34, 36]:

$$X_{n+1} + X_{n-1} = \frac{(an+b)X_n + c}{1 - X_n{}^2}. \tag{3.10}$$

Equation (3.10) is referred to as a discrete Painlevé II equation, denoted by d-$\mathrm{P_{II}}$, and has the space of initial condition of type $D_5^{(1)}$. d-$\mathrm{P_{II}}$ admits a specialization to the discrete Riccati equation:

$$X_{n+1} = \frac{4X_n - 2an - a - 2b + 4}{4(X_n + 1)}, \tag{3.11}$$

with

$$c = -\frac{a}{2}. \tag{3.12}$$

Therefore we obtain the following theorem:

Theorem 3.1. d-$\mathrm{P_{II}}$ (3.10) *admits the following solution*:

$$X_n = \frac{2(n+1)}{c_2(c_1 + c_0 c_2)} x_n + 1, \quad (n \in \mathbb{Z}_{\geq 0}). \tag{3.13}$$

Here

$$a = \frac{8}{(c_1 + c_0 c_2)^2}, \quad b = \frac{12}{(c_1 + c_0 c_2)^2}, \quad c = -\frac{4}{(c_1 + c_0 c_2)^2}, \quad c_1 + c_0 c_2 \neq 0. \tag{3.14}$$

3.2 Example II: The case where $(\hat{P}_n)_{n=0}^{\infty}$ are the Hermite polynomials

We next consider the case where $\hat{P}_n$ is the Hermite polynomial:

$$\hat{P}_n(t) = \frac{H_n(c_2 t + c_1)}{{c_2}^n}, \quad (c_2 > 0,\ c_1 \in \mathbb{C}). \tag{3.15}$$

We assume here that c_0 is not a real number. The linear functionals are given as

$$\mathcal{L}[f(t)] = \int_{-\infty}^{\infty} \frac{f(t)}{t - c_0}\, e^{-{c_2}^2 t^2/2 - c_1 c_2 t} \mathrm{d}t, \tag{3.16}$$

$$\hat{\mathcal{L}}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-{c_2}^2 t^2/2 - c_1 c_2 t} \mathrm{d}t, \tag{3.17}$$

and the moments are given by the following lemma:

Lemma 3.1. *The following equations hold*:

$$\hat{\mu}_n = \frac{2^{1/2}\sqrt{\pi}}{{c_2}^{n+1}}\, e^{n\pi i/2 + {c_1}^2/4} D_n(ic_1), \tag{3.18}$$

$$\mu_n = c_0 \mu_{n-1} + \hat{\mu}_{n-1}, \tag{3.19}$$

$$\mu_0 = \begin{cases} -\pi i e^{-c_0 c_1 c_2 - {c_0}^2 {c_2}^2/2} & (\mathrm{Im}(c_0) > 0), \\ \pi i e^{-c_0 c_1 c_2 - {c_0}^2 {c_2}^2/2} & (\mathrm{Im}(c_0) < 0), \end{cases} \tag{3.20}$$

where $D_\lambda(z)$ is the parabolic cylinder function.

Proof. Equations (3.18) and (3.19) are obvious. We prove (3.20). Replacing t with $\sqrt{2}{c_2}^{-1}t + c_0$, we can rewrite μ_0 as

$$\mu_0 = e^{-c_0 c_1 c_2 - {c_0}^2 {c_2}^2/2} \int_{-\infty + c_0}^{\infty + c_0} \frac{e^{-t^2 - \sqrt{2}(c_0 c_2 + c_1)t}}{t}\, \mathrm{d}t. \tag{3.21}$$

Using the integral representation of the Legendre polynomial:

$$P_n(z) = \frac{i}{\pi} \int_C \frac{e^{-t^2 + 2tz}}{t^{n+1}}\, \mathrm{d}t, \tag{3.22}$$

where the contour C runs from $-\infty$ to $+\infty$ so that $t = 0$ lie to the right of the contour, we obtain

$$\mu_0 = \begin{cases} -\pi i e^{-c_0 c_1 c_2 - {c_0}^2 {c_2}^2/2} P_0\left(-\dfrac{c_0 c_2 + c_1}{\sqrt{2}}\right) & (\mathrm{Im}(c_0) > 0), \\ -\pi i e^{-c_0 c_1 c_2 - {c_0}^2 {c_2}^2/2} \left(P_0\left(-\dfrac{c_0 c_2 + c_1}{\sqrt{2}}\right) - 2\right) & (\mathrm{Im}(c_0) < 0). \end{cases} \tag{3.23}$$

Then the statement follows from

$$P_0(z) = 1. \tag{3.24}$$

■

From the three-term recurrence relation:

$$t\hat{P}_n = \hat{P}_{n+1} - \frac{c_1}{c_2}\,\hat{P}_n + \frac{n}{{c_2}^2}\,\hat{P}_{n-1}, \tag{3.25}$$

we obtain

$$\hat{\alpha}_n = -\frac{c_1}{c_2}\,, \quad \hat{\beta}_n = \frac{n}{{c_2}^2}\,. \tag{3.26}$$

From (2.21), we obtain the following discrete Riccati equation:

$$y_n = -\frac{{c_2}^2}{ny_{n-1} + c_2(c_1 + c_0c_2)}\,. \tag{3.27}$$

Therefore we obtain the following theorem:

Theorem 3.2. d-$\mathrm{P_{II}}$ (3.10) *admits the following solution*:

$$X_n = \frac{2(n+1)}{c_2(c_1 + c_0c_2)}\,y_n + 1, \quad (n \in \mathbb{Z}_{\geq 0}). \tag{3.28}$$

Here

$$a = \frac{8}{(c_1 + c_0c_2)^2}\,, \quad b = \frac{12}{(c_1 + c_0c_2)^2}\,, \quad c = -\frac{4}{(c_1 + c_0c_2)^2}\,, \quad c_1 + c_0c_2 \neq 0. \tag{3.29}$$

3.3 Example III: The case where $(P_n)_{n=0}^{\infty}$ are the discrete q-Hermite II polynomials

We define P_n as

$$P_n(t) = \frac{h_n^{\mathrm{II}}(c_1t;q)}{{c_1}^n}\,, \quad (c_1 > 0), \tag{3.30}$$

where h_n^{II} is the discrete q-Hermite II polynomial:

$$h_n^{\mathrm{II}}(t;q) = t^n\,{}_2\varphi_1\left(\begin{matrix} q^{-n}, q^{-n+1} \\ 0 \end{matrix}; q^2, -\frac{q^2}{t^2}\right). \tag{3.31}$$

The linear functionals, the three-term recurrence relation and the discrete Riccati equation are given by

$$\mathcal{L}[f(t)] = \int_{-\infty}^{\infty} \frac{f(t)}{(-{c_1}^2t^2;q^2)_\infty}\,\mathrm{d}_qt, \tag{3.32}$$

$$\hat{\mathcal{L}}[f(t)] = \int_{-\infty}^{\infty} \frac{(t-c_0)f(t)}{(-{c_1}^2t^2;q^2)_\infty}\,\mathrm{d}_qt, \tag{3.33}$$

$$tP_n = P_{n+1} + q^{-2n+1}(1-q^n){c_1}^{-2}P_{n-1}, \tag{3.34}$$

$$x_n = -\frac{1}{q^{-2n+1}(1-q^n){c_1}^{-2}x_{n-1} + c_0}\,, \tag{3.35}$$

respectively. For the moment sequences, the following lemma holds:

Lemma 3.2. *The following equations hold*:

$$\mu_n = \begin{cases} 2\,\dfrac{(q^2;q^2)_\infty}{(q^3;q^2)_\infty}\,\dfrac{\Theta(-q{c_1}^2;q^2)}{\Theta(-{c_1}^2;q^2)}\,\dfrac{(q;q^2)_k}{q^{k^2}{c_1}^{2k}} & (n=2k),\\ 0 & (n=2k+1), \end{cases} \tag{3.36}$$

$$\hat{\mu}_n = \mu_{n+1} - c_0\mu_n. \tag{3.37}$$

Proof. Equation (3.37) is obvious. We prove (3.36). From

$$\mu_{2k-2} = \int_{-\infty}^{\infty} \frac{(1+q^{-2}{c_1}^2t^2)t^{2k-2}}{(-q^{-2}{c_1}^2t^2;q^2)_\infty}\,\mathrm{d}_qt \tag{3.38}$$

$$= q^{2k-1}\int_{-\infty}^{\infty} \frac{(1+{c_1}^2t^2)t^{2k-2}}{(-{c_1}^2t^2;q^2)_\infty}\,\mathrm{d}_qt \tag{3.39}$$

$$= q^{2k-1}\mu_{2k-2} + q^{2k-1}{c_1}^2\mu_{2k}, \tag{3.40}$$

we obtain

$$\mu_{2k} = \frac{1-q^{2k-1}}{q^{2k-1}{c_1}^2}\,\mu_{2k-2} = \cdots = \frac{(q;q^2)_k}{q^{k^2}{c_1}^{2k}}\,\mu_0. \tag{3.41}$$

From

$$h_n = 2\,\frac{(q^2;q^2)_\infty}{(q^3;q^2)_\infty}\,\frac{\Theta(-q{c_1}^2;q^2)}{\Theta(-{c_1}^2;q^2)}\,\frac{(q;q)_n}{q^{n^2}{c_1}^{2n}}\,, \tag{3.42}$$

we obtain

$$\mu_0 = h_0 = 2\,\frac{(q^2;q^2)_\infty}{(q^3;q^2)_\infty}\,\frac{\Theta(-q{c_1}^2;q^2)}{\Theta(-{c_1}^2;q^2)}\,. \tag{3.43}$$

Therefore we have completed the proof. ∎

We obtain the following theorem:

Theorem 3.3. q-$\mathrm{P_{IV}}$ (1.7) *admits the following solution*:

$$X_n = i\,\frac{1-q^{n+1}}{c_1q^n}\,x_n. \tag{3.44}$$

Here

$${a_0}^{1/2} = -iq^{-m+1}{c_0}^{-1}{c_1}^{-1}, \quad {a_0}^{1/2}{a_1}^{1/2} = q^{-N}, \quad a_2 = q^{2N+2}, \quad c_0 \neq 0. \tag{3.45}$$

We find that the solution to q-$\mathrm{P_{IV}}$ given in Theorem 3.3 coincides with one given in (1.29).

3.4 Example IV: The case where $(\hat{P}_n)_{n=0}^{\infty}$ are the discrete q-Hermite II polynomials

We consider the case where $\hat{P}_n$ is the discrete q-Hermite II polynomial:

$$\hat{P}_n(t) = \frac{h_n^{\mathrm{II}}(c_1t;q)}{{c_1}^n}\,, \quad (c_1>0). \tag{3.46}$$

We assume that $c_0 \neq q^a$ for $^\forall a \in \mathbb{Z}$. Then we have the linear functionals and the moments as

$$\mathcal{L}[f(t)] = \int_{-\infty}^{\infty} \frac{f(t)}{(t-c_0)(-c_1{}^2t^2;q^2)_\infty}\,\mathrm{d}_qt, \tag{3.47}$$

$$\hat{\mathcal{L}}[f(t)] = \int_{-\infty}^{\infty} \frac{f(t)}{(-c_1{}^2t^2;q^2)_\infty}\,\mathrm{d}_qt, \tag{3.48}$$

$$\mu_n = 2\,\frac{(1-q)(1-c_0{}^2)(-c_1{}^2;q^2)_\infty}{c_0{}^3}\,{}_2\psi_2\left(\begin{matrix}0, q^2c_0{}^{-2}\\ -c_1{}^2, c_0{}^{-2}\end{matrix}; q^2, q^{n+1}\right), \tag{3.49}$$

$$\hat{\mu}_n = \begin{cases} 2\,\dfrac{(q^2;q^2)_\infty}{(q^3;q^2)_\infty}\,\dfrac{\Theta(-qc_1{}^2;q^2)}{\Theta(-c_1{}^2;q^2)}\,\dfrac{(q;q^2)_k}{q^{k^2}c_1{}^{2k}} & (n=2k),\\ 0 & (n=2k+1). \end{cases} \tag{3.50}$$

Three-term recurrence relation and the discrete Riccati equation are given by

$$t\hat{P}_n = \hat{P}_{n+1} + q^{-2n+1}(1-q^n)c_1{}^{-2}\hat{P}_{n-1}, \tag{3.51}$$

$$y_n = -\frac{1}{q^{-2n+1}(1-q^n)c_1{}^{-2}y_{n-1}+c_0}. \tag{3.52}$$

Theorem 3.4. q-$\mathrm{P_{IV}}$ (1.7) *admits the following solution*:

$$X_n = i\,\frac{1-q^{n+1}}{c_1q^n}\,y_n. \tag{3.53}$$

Here

$$a_0{}^{1/2} = -iq^{-m+1}c_0{}^{-1}c_1{}^{-1}, \quad a_0{}^{1/2}a_1{}^{1/2} = q^{-N}, \quad a_2 = q^{2N+2}, \quad c_0 \neq 0. \tag{3.54}$$

4 Concluding remarks

In this paper, we constructed the method to give the solutions to discrete Painlevé systems expressed in terms of normalization factors of two types of orthogonal polynomials and also presented some examples.

It seems that the solutions of various discrete Painlevé systems can be constructed by using the method in this paper. One interesting project is to make a list of discrete Painlevé systems related with orthogonal polynomials given in [19] by this method.

Before closing this paper, we briefly discuss the structure of the solutions of discrete Painlevé systems derived in this paper. It is well known that the τ functions play a crucial role in the theory of integrable systems including Painlevé systems[13, 14, 15, 18, 22, 27, 28, 30, 31, 32, 33]. Further, it is also known that the particular solutions to Painlevé systems are expressible in the form of ratio of determinants, and the determinants directly related with τ functions[20, 21, 23, 25]. However we have not yet clarified the relation between the solutions obtained in this paper and the τ functions. For example, let us consider the q-$\mathrm{P_{IV}}$ (1.7). In [40], T. Tsuda introduced the τ functions for q-Painlevé equations, including q-$\mathrm{P_{IV}}$ (1.7), with affine Weyl group symmetry of type $A_4^{(1)}$. The solutions given in the following proposition are constructed by the method to construct the hypergeometric τ functions (cf. [20, 21, 23, 25]). Therefore the determinants of this solution are directly related to the τ functions.

Proposition 4.1 ([24])**.** *When* $a_0{}^{1/2}a_1{}^{1/2} = q$ *and* $N \geq 0$, q-$\mathrm{P_{IV}}$ (1.7) *has the following solutions*:

$$X_n = X_n(m,N) = -q^{-2N-m+1}a_0{}^{-1/2}\,\frac{\phi_N^{n,m}\,\phi_{N+1}^{n+3,m+1}}{\phi_{N+1}^{n+2,m+1}\,\phi_N^{n+1,m}}, \tag{4.1}$$

where

$$\phi_0^{n,m} = 1, \tag{4.2}$$

$$\phi_N^{n,m} = \begin{vmatrix} F_{n,m} & F_{n+1,m} & \cdots & F_{n+N-1,m} \\ F_{n-2,m} & F_{n-1,m} & \cdots & F_{n+N-3,m} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n-2N+2,m} & F_{n-2N+3,m} & \cdots & F_{n-N+1,m} \end{vmatrix} \quad (N \in \mathbb{Z}_{>0}). \tag{4.3}$$

Here, $F_{n,m}$ is given as

$$\begin{aligned} F_{n,m} = {} & A_{n,m} \, \frac{\Theta(q^n a_2; q)\Theta(q^{2m-1}a_0; q^2)(q^{m-1}{a_0}^{1/2}; q)_\infty}{\Theta(q^{n+m-2}{a_0}^{1/2}a_2; q)} \\ & \times {}_2\varphi_1\left(\begin{matrix} 0, q^{-m+2}{a_0}^{-1/2} \\ -q \end{matrix}; q, q^{n-1}a_2\right) \\ & + B_{n,m} \, \frac{\Theta(q^n a_2; q)\Theta(-q^{2m-1}a_0; q^2)(-q^{m-1}{a_0}^{1/2}; q)_\infty}{\Theta(-q^{n+m-2}{a_0}^{1/2}a_2; q)} \\ & \times {}_2\varphi_1\left(\begin{matrix} 0, -q^{-m+2}{a_0}^{-1/2} \\ -q \end{matrix}; q, q^{n-1}a_2\right), \end{aligned} \tag{4.4}$$

where $A_{n,m}$ and $B_{n,m}$ are periodic functions of period one for n and m, i.e.,

$$A_{n,m} = A_{n+1,m} = A_{n,m+1}, \quad B_{n,m} = B_{n+1,m} = B_{n,m+1}. \tag{4.5}$$

Comparing the configuration of $\psi_n^{l,m}$ in (1.8) with one of $\phi_N^{n,m}$ in (4.1), we find that the relation between the determinants of the solutions given in this paper and the τ functions is not obvious. This point will be investigated in forthcoming paper [24]. We note here that the solutions given in this paper are called molecule type solutions, whose determinant size depends on an independent variable and ones given in Proposition 4.1 are called lattice type solutions, whose determinant size does not depend on an independent variable.

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A Proof of Lemma 1.1

In this appendix, we prove Lemma 1.1. We first consider the determinant expression of $\psi_n^{l,m}$.

Lemma A.1. *$\psi_n^{l,m}$ can be rewritten as*

$$\psi_n^{l,m} = \left| \left\{ q^{-(i-1)(i+j-2)} I_{l+i+j-2,m+i-1} \right\}_{1\le i,j\le n} \right| \tag{A.1}$$

$$= \begin{vmatrix} I_{l,m} & q^{-1}I_{l+1,m+1} & \cdots & q^{-(n-1)^2}I_{l+n-1,m+n-1} \\ I_{l+1,m} & q^{-2}I_{l+2,m+1} & \cdots & q^{-n(n-1)}I_{l+n,m+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{l+n-1,m} & q^{-n}I_{l+n,m+1} & \cdots & q^{-2(n-1)^2}I_{l+2n-2,m+n-1} \end{vmatrix}. \tag{A.2}$$

Here

$$I_{l,m} = \int_{-\infty}^{\infty} \frac{\prod_{k=0}^{l-1}(q^k t - q^m a)}{E_{q^2}\left({c_1}^2 t^2\right)} \, \mathrm{d}_q t. \tag{A.3}$$

Proof. The statement follows from the following calculation:

$$\left|\left\{q^{-(i-1)(i+j-2)}I_{l+i+j-2,m+i-1}\right\}_{1\le i,j\le n}\right| \tag{A.4}$$

$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{i=1}^{n}\frac{\mathrm{d}_qt_i}{E_{q^2}\left({c_1}^2{t_i}^2\right)}\right)$$
$$\times\begin{vmatrix}\prod_{k=0}^{l-1}(q^kt_1-q^ma) & \frac{\prod_{k=0}^{l}(q^{k-1}t_2-q^ma)}{q^{-l}} & \cdots & \frac{\prod_{k=0}^{l+n-2}(q^{k-n+1}t_n-q^ma)}{q^{-(n-1)l}}\\ \prod_{k=0}^{l}(q^kt_1-q^ma) & \frac{\prod_{k=0}^{l+1}(q^{k-1}t_2-q^ma)}{q^{-l}} & \cdots & \frac{\prod_{k=0}^{l+n-1}(q^{k-n+1}t_n-q^ma)}{q^{-(n-1)l}}\\ \vdots & \vdots & \ddots & \vdots\\ \prod_{k=0}^{l+n-2}(q^kt_1-q^ma) & \frac{\prod_{k=0}^{l+n-1}(q^{k-1}t_2-q^ma)}{q^{-l}} & \cdots & \frac{\prod_{k=0}^{l+2n-3}(q^{k-n+1}t_n-q^na)}{q^{-(n-1)l}}\end{vmatrix} \tag{A.5}$$

$$=q^{n(n-1)l/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{i=1}^{n}\frac{\prod_{k=0}^{l+i-2}(q^{k-i+1}t_i-q^ma)}{E_{q^2}\left({c_1}^2{t_i}^2\right)}\mathrm{d}_qt_i\right)$$
$$\times\begin{vmatrix}1 & 1 & \cdots & 1\\ \prod_{k=0}^{0}(q^{k+l}t_1-q^ma) & \prod_{k=0}^{1}(q^{k+l-1}t_2-q^ma) & \cdots & \prod_{k=0}^{n-1}(q^{k+l-n+1}t_n-q^ma)\\ \vdots & \vdots & \ddots & \vdots\\ \prod_{k=0}^{n-2}(q^{k+l}t_1-q^ma) & \prod_{k=0}^{n-1}(q^{k+l-1}t_2-q^ma) & \cdots & \prod_{k=0}^{2n-3}(q^{k+l-n+1}t_n-q^na)\end{vmatrix} \tag{A.6}$$

$$=q^{n(n-1)(n+6l-2)/6}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Delta(t_1,\cdots,t_n)\prod_{i=1}^{n}\frac{\prod_{k=0}^{l+i-2}(q^{k-i+1}t_i-q^ma)}{E_{q^2}\left({c_1}^2{t_i}^2\right)}\mathrm{d}_qt_i \tag{A.7}$$

$$=\frac{q^{n(n-1)(n+6l-2)/6}}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Delta(t_1,\cdots,t_n)$$
$$\times\sum_{\sigma\in\mathfrak{S}_n}(\mathrm{sgn}\ \sigma)\prod_{i=1}^{n}\frac{\prod_{k=0}^{l+i-2}(q^{k-i+1}t_{\sigma(i)}-q^ma)}{E_{q^2}\left((1-q^2)q^{-3}{t_{\sigma(i)}}^2\right)}\mathrm{d}_qt_{\sigma(i)} \tag{A.8}$$

$$=\frac{q^{n(n-1)(n+6l-2)/6}}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Delta(t_1,\cdots,t_n)\left(\prod_{i=1}^{n}\frac{\mathrm{d}_qt_i}{E_{q^2}\left({c_1}^2{t_i}^2\right)}\right)$$
$$\times\begin{vmatrix}\prod_{k=0}^{l-1}(q^kt_1-q^ma) & \prod_{k=0}^{l-1}(q^kt_2-q^ma) & \cdots & \prod_{k=0}^{l-1}(q^kt_n-q^ma)\\ \prod_{k=0}^{l}(q^{k-1}t_1-q^ma) & \prod_{k=0}^{l}(q^{k-1}t_2-q^ma) & \cdots & \prod_{k=0}^{l}(q^{k-1}t_n-q^ma)\\ \vdots & \vdots & \ddots & \vdots\\ \prod_{k=0}^{l+n-2}(q^{k-n+1}t_1-q^ma) & \prod_{k=0}^{l+n-2}(q^{k-n+1}t_2-q^ma) & \cdots & \prod_{k=0}^{l+n-2}(q^{k-n+1}t_n-q^ma)\end{vmatrix} \tag{A.9}$$

$$=\frac{q^{n(n-1)(n+6l-2)/6}}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Delta(t_1,\cdots,t_n)\left(\prod_{i=1}^{n}\frac{\mathrm{d}_qt_i}{E_{q^2}\left({c_1}^2{t_i}^2\right)}\right)$$
$$\times\begin{vmatrix}\prod_{k=0}^{l-1}(q^{l-1-k}t_1-q^ma) & \prod_{k=0}^{l-1}(q^{l-1-k}t_2-q^ma) & \cdots & \prod_{k=0}^{l-1}(q^{l-1-k}t_n-q^ma)\\ \prod_{k=0}^{l}(q^{l-1-k}t_1-q^ma) & \prod_{k=0}^{l}(q^{l-1-k}t_2-q^ma) & \cdots & \prod_{k=0}^{l}(q^{l-1-k}t_n-q^ma)\\ \vdots & \vdots & \ddots & \vdots\\ \prod_{k=0}^{l+n-2}(q^{l-1-k}t_1-q^ma) & \prod_{k=0}^{l+n-2}(q^{l-1-k}t_2-q^ma) & \cdots & \prod_{k=0}^{l+n-2}(q^{l-1-k}t_n-q^ma)\end{vmatrix} \tag{A.10}$$

$$=\frac{q^{n(n-1)(n+6l-2)/6}}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Delta(t_1,\cdots,t_n)\left(\prod_{i=1}^{n}\frac{\prod_{k=0}^{l-1}(q^kt_i-q^ma)}{E_{q^2}\left({c_1}^2{t_i}^2\right)}\,\mathrm{d}_qt_i\right)$$

$$\times\begin{vmatrix}1 & 1 & \cdots & 1\\ \prod_{k=0}^{0}(q^{-1-k}t_1-q^ma) & \prod_{k=0}^{0}(q^{-1-k}t_2-q^ma) & \cdots & \prod_{k=0}^{0}(q^{-1-k}t_n-q^ma)\\ \vdots & \vdots & \ddots & \vdots\\ \prod_{k=0}^{n-2}(q^{-1-k}t_1-q^ma) & \prod_{k=0}^{n-2}(q^{-1-k}t_2-q^ma) & \cdots & \prod_{k=0}^{n-2}(q^{-1-k}t_n-q^ma)\end{vmatrix} \tag{A.11}$$

$$=\frac{q^{n(n-1)(2l-1)/2}}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Delta(t_1,\cdots,t_n)^2\prod_{i=1}^{n}\frac{\prod_{k=0}^{l-1}(q^kt_i-q^ma)}{E_{q^2}\left({c_1}^2{t_i}^2\right)}\,\mathrm{d}_qt_i. \tag{A.12}$$

■

We next consider the linear relations of $I_{n,m}$.

Lemma A.2. *The following contiguity relations hold*:

$$I_{l+1,m+1}-I_{l+1,m}-(1-q^{l+1})q^maI_{l,m}=0, \tag{A.13}$$

$$I_{l+2,m}+q^maI_{l+1,m}-q^l\frac{1-q^{l+1}}{{c_1}^2}\,I_{l,m}=0, \tag{A.14}$$

$$(1+a^2{c_1}^2q^{2m+2})I_{l,m}+a{c_1}^2q^{-l+m+1}I_{l+1,m+1}-I_{l,m+1}=0. \tag{A.15}$$

Proof. Equation (A.13) follows immediately from (A.3). We next prove (A.14). Using the partial integration (1.42), we obtain

$$\int_{-\infty}^{\infty}t\frac{\prod_{k=0}^{l-1}(q^kt-q^ma)}{E_{q^2}\left({c_1}^2t^2\right)}\,\mathrm{d}_qt=\frac{q-1}{{c_1}^2}\int_{-\infty}^{\infty}\left(\prod_{k=0}^{l-1}(q^kt-q^ma)\right)D_{q,t}\left(\frac{1}{E_{q^2}\left({c_1}^2t^2\right)}\right)\mathrm{d}_qt \tag{A.16}$$

$$=\frac{1-q^l}{q{c_1}^2}\,I_{l-1,m}. \tag{A.17}$$

Therefore (A.14) is derived as follows:

$$I_{l+1,m}=\int_{-\infty}^{\infty}\frac{\prod_{k=0}^{l}(q^kt-q^ma)}{E_{q^2}\left({c_1}^2t^2\right)}\,\mathrm{d}_qt=q^{l-1}\frac{1-q^l}{{c_1}^2}\,I_{l-1,m}-q^maI_{l,m}. \tag{A.18}$$

Finally, we show (A.15). Eliminating $I_{l,m}$ from (A.13) and (A.14), we obtain

$$I_{l,m+1}-(1+q^{-l+2m+1}a^2{c_1}^2)I_{l,m}-a{c_1}^2q^{-l+m+1}I_{l+1,m}=0. \tag{A.19}$$

Further, eliminating $I_{l+1,m}$ from (A.19) and (A.13), we obtain (A.15). ■

Using the contiguity relations (A.13)–(A.15), we construct the bilinear equations of $\psi_n^{l,m}$. Let $\boldsymbol{a}_k^{l,m}$, $\boldsymbol{b}_k^{h,l,m}$ and $\boldsymbol{e}_k$ be k-dimensional vectors given as

$$\boldsymbol{a}_k^{l,m}=\begin{pmatrix}(1-q^{l+1})I_{l,m}\\(1-q^{l+2})I_{l+1,m}\\ \vdots\\(1-q^{l+k})I_{l+k-1,m}\end{pmatrix},\quad \boldsymbol{b}_k^{h,l,m}=\begin{pmatrix}q^{-(h-1)^2}I_{l,m}\\q^{-h(h-1)}I_{l+1,m}\\ \vdots\\q^{-(h+k-2)(h-1)}I_{l+k-1,m}\end{pmatrix},\quad \boldsymbol{e}_k=\begin{pmatrix}0\\0\\ \vdots\\1\end{pmatrix}, \tag{A.20}$$

respectively. We also introduce the following notation:

$$\left|\boldsymbol{v}_k^1,\boldsymbol{v}_k^2,\ldots,\boldsymbol{v}_k^k\right|=\begin{vmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,k}\\ v_{2,1} & v_{2,2} & \cdots & v_{2,k}\\ \vdots & \vdots & \ddots & \vdots\\ v_{k,1} & v_{k,2} & \cdots & v_{k,k}\end{vmatrix},\tag{A.21}$$

for arbitrary k-dimensional vector

$$\boldsymbol{v}_k^l=\begin{pmatrix} v_{1,l}\\ v_{2,l}\\ \vdots\\ v_{k,l}\end{pmatrix}.\tag{A.22}$$

It is easy to verify the following equalities from the definition:

$$\left|\boldsymbol{a}_n^{l,m},\boldsymbol{a}_n^{l,m+1},\cdots,\boldsymbol{a}_n^{l,m+n-1}\right|=(q^{l+1};q)_n\left|\boldsymbol{b}_n^{1,l,m},\boldsymbol{b}_n^{1,l,m+1},\cdots,\boldsymbol{b}_n^{1,l,m+n-1}\right|,\tag{A.23}$$

$$\left|\boldsymbol{a}_n^{l,m},\boldsymbol{a}_n^{l,m+1},\cdots,\boldsymbol{a}_n^{l,m+n-2},\boldsymbol{e}_n\right|=(q^{l+1};q)_{n-1}\left|\boldsymbol{b}_{n-1}^{1,l,m},\boldsymbol{b}_{n-1}^{1,l,m+1},\cdots,\boldsymbol{b}_{n-1}^{1,l,m+n-2}\right|,\tag{A.24}$$

$$\left|\boldsymbol{b}_n^{2,l,m},\boldsymbol{b}_n^{2,l,m+1},\cdots,\boldsymbol{b}_n^{2,l,m+n-1}\right|=q^{-n(n+1)/2}\left|\boldsymbol{b}_n^{1,l,m},\boldsymbol{b}_n^{1,l,m+1},\cdots,\boldsymbol{b}_n^{1,l,m+n-1}\right|,\tag{A.25}$$

$$\left|\boldsymbol{b}_n^{2,l,m},\boldsymbol{b}_n^{2,l,m+1},\cdots,\boldsymbol{b}_n^{2,l,m+n-2},\boldsymbol{e}_n\right|=q^{-n(n-1)/2}\left|\boldsymbol{b}_{n-1}^{1,l,m},\boldsymbol{b}_{n-1}^{1,l,m+1},\cdots,\boldsymbol{b}_{n-1}^{1,l,m+n-2}\right|.\tag{A.26}$$

Note that (A.13) implies the following relation:

$$\boldsymbol{a}_k^{l,m}=a^{-1}q^{-m}\boldsymbol{b}_k^{1,l+1,m+1}-a^{-1}q^{-m}\boldsymbol{b}_k^{1,l+1,m}.\tag{A.27}$$

Then we obtain the following lemma:

Lemma A.3. *It holds that*

$$\begin{aligned}&\left|\boldsymbol{a}_n^{l,m+1},\boldsymbol{a}_n^{l,m+2},\cdots,\boldsymbol{a}_n^{l,m+n-2},\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{e}_n\right|\\ &\quad=(-1)^n a^{-n+2}q^{-(n-2)(2m+n-1)/2}\left|\boldsymbol{b}_{n-1}^{1,l+1,m+1},\boldsymbol{b}_{n-1}^{1,l+1,m+2},\cdots,\boldsymbol{b}_{n-1}^{1,l+1,m+n-1}\right|,\end{aligned}\tag{A.28}$$

$$\begin{aligned}&\left|\boldsymbol{a}_n^{l,m},\boldsymbol{a}_n^{l,m+1},\cdots,\boldsymbol{a}_n^{l,m+n-2},\boldsymbol{b}_n^{1,l+1,m+1}\right|\\ &\quad=(-1)^{n-1} a^{-n+1}q^{-(n-1)(2m+n-2)/2}\left|\boldsymbol{b}_n^{1,l+1,m},\boldsymbol{b}_n^{1,l+1,m+1},\cdots,\boldsymbol{b}_n^{1,l+1,m+n-1}\right|,\end{aligned}\tag{A.29}$$

$$\begin{aligned}&\left|\boldsymbol{a}_n^{l,m+1},\boldsymbol{a}_n^{l,m+2},\cdots,\boldsymbol{a}_n^{l,m+n-1},\boldsymbol{b}_n^{1,l+1,m+1}\right|\\ &\quad=(-1)^{n-1} a^{-n+1}q^{-(n-1)(2m+n)/2}\left|\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{b}_n^{1,l+1,m+2},\cdots,\boldsymbol{b}_n^{1,l+1,m+n}\right|.\end{aligned}\tag{A.30}$$

Proof. We first prove (A.28). From (A.27), we have

$$\begin{aligned}&\left|\boldsymbol{a}_n^{l,m+1},\boldsymbol{a}_n^{l,m+2},\cdots,\boldsymbol{a}_n^{l,m+n-2},\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{e}_n\right|\\ &\quad=(-1)^n\left|\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{a}_n^{l,m+1},\boldsymbol{a}_n^{l,m+2},\cdots,\boldsymbol{a}_n^{l,m+n-2},\boldsymbol{e}_n\right| &\text{(A.31)}\\ &\quad=(-1)^n a^{-1}q^{-m-1}\left|\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{b}_n^{1,l+1,m+2},\boldsymbol{a}_n^{l,m+2},\cdots,\boldsymbol{a}_n^{l,m+n-2},\boldsymbol{e}_n\right|. &\text{(A.32)}\end{aligned}$$

Repeating this procedure, we obtain

$$\begin{aligned}&\left|\boldsymbol{a}_n^{l,m+1},\boldsymbol{a}_n^{l,m+2},\cdots,\boldsymbol{a}_n^{l,m+n-2},\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{e}_n\right|\\ &\quad=(-1)^n a^{-n+2}q^{-(n-2)(2m+n-1)/2}\left|\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{b}_n^{1,l+1,m+2},\cdots,\boldsymbol{b}_n^{1,l+1,m+n-1},\boldsymbol{e}_n\right| &\text{(A.33)}\\ &\quad=(-1)^n a^{-n+2}q^{-(n-2)(2m+n-1)/2}\left|\boldsymbol{b}_{n-1}^{1,l+1,m+1},\boldsymbol{b}_{n-1}^{1,l+1,m+2},\cdots,\boldsymbol{b}_{n-1}^{1,l+1,m+n-1}\right|. &\text{(A.34)}\end{aligned}$$

We next show (A.29). From (A.27), we have

$$\begin{aligned}
&\left|\boldsymbol{a}_n^{l,m},\boldsymbol{a}_n^{l,m+1},\cdots,\boldsymbol{a}_n^{l,m+n-2},\boldsymbol{b}_n^{1,l+1,m+1}\right|\\
&\quad=(-1)^n\left|\boldsymbol{a}_n^{l,m},\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{a}_n^{l,m+1},\cdots,\boldsymbol{a}_n^{l,m+n-2}\right| &(\mathrm{A.35})\\
&\quad=(-1)^{n+1}a^{-1}q^{-m}\left|\boldsymbol{b}_n^{1,l+1,m},\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{a}_n^{l,m+1},\boldsymbol{a}_n^{l,m+2},\cdots,\boldsymbol{a}_n^{l,m+n-2}\right| &(\mathrm{A.36})\\
&\quad=(-1)^{n+1}a^{-2}q^{-2m-1}\left|\boldsymbol{b}_n^{1,l+1,m},\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{b}_n^{1,l+1,m+2},\boldsymbol{a}_n^{l,m+2},\boldsymbol{a}_n^{l,m+3},\cdots,\boldsymbol{a}_n^{l,m+n-2}\right|. &(\mathrm{A.37})
\end{aligned}$$

Repeating this procedure, we obtain (A.29).

Finally, (A.30) can be derived in a similar manner by using (A.27) as

$$\begin{aligned}
&\left|\boldsymbol{a}_n^{l,m+1},\boldsymbol{a}_n^{l,m+2},\cdots,\boldsymbol{a}_n^{l,m+n-1},\boldsymbol{b}_n^{1,l+1,m+1}\right|\\
&\quad=(-1)^{n-1}\left|\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{a}_n^{l,m+1},\boldsymbol{a}_n^{l,m+2},\cdots,\boldsymbol{a}_n^{l,m+n-1}\right| &(\mathrm{A.38})\\
&\quad=(-1)^{n-1}a^{-1}q^{-m-1}\left|\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{b}_n^{1,l+1,m+2},\boldsymbol{a}_n^{l,m+2},\boldsymbol{a}_n^{l,m+3},\cdots,\boldsymbol{a}_n^{l,m+n-1}\right|, &(\mathrm{A.39})
\end{aligned}$$

which completes the proof. ■

Similarly, we have the following equation from (A.15):

$$\boldsymbol{b}_k^{h+1,l+1,m+1}=\frac{1}{{c_1}^2aq^{-l+m+2h}}\,\boldsymbol{b}_k^{h,l,m+1}-\frac{1+{c_1}^2a^2q^{2m+2}}{{c_1}^2aq^{-l+m+2h}}\,\boldsymbol{b}_k^{h,l,m}, \tag{A.40}$$

which yields the following lemma:

Lemma A.4. *It holds that*

$$\begin{aligned}
&\left|\boldsymbol{b}_n^{2,l+1,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\cdots,\boldsymbol{b}_n^{2,l+1,m+n-1},\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{e}_n\right|\\
&\quad=(-1)^n\left(\prod_{k=1}^{n-2}\frac{1}{{c_1}^2aq^{-l+m+k+2}}\right)\left|\boldsymbol{b}_{n-1}^{1,l,m+1},\boldsymbol{b}_{n-1}^{1,l,m+2},\cdots,\boldsymbol{b}_{n-1}^{1,l,m+n-1}\right|, &(\mathrm{A.41})\\
&\left|\boldsymbol{b}_n^{2,l+1,m+1},\boldsymbol{b}_n^{2,l+1,m+2},\cdots,\boldsymbol{b}_n^{2,l+1,m+n-1},\boldsymbol{b}_n^{1,l,m+1}\right|\\
&\quad=(-1)^{n+1}\frac{1+{c_1}^2a^2q^{2m+2}}{{c_1}^2aq^{-l+m+2}}\left(\prod_{k=1}^{n-2}\frac{1}{{c_1}^2aq^{-l+m+k+2}}\right)\left|\boldsymbol{b}_n^{1,l,m},\boldsymbol{b}_n^{1,l,m+1},\cdots,\boldsymbol{b}_n^{1,l,m+n-1}\right|, &(\mathrm{A.42})\\
&\left|\boldsymbol{b}_n^{2,l+1,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\cdots,\boldsymbol{b}_n^{2,l+1,m+n},\boldsymbol{b}_n^{1,l,m+1}\right|\\
&\quad=(-1)^{n-1}\left(\prod_{k=1}^{n-1}\frac{1}{{c_1}^2aq^{-l+m+k+2}}\right)\left|\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{b}_n^{1,l,m+2},\cdots,\boldsymbol{b}_n^{1,l,m+n}\right|. &(\mathrm{A.43})
\end{aligned}$$

Proof. We first prove (A.41). From (A.40), we have

$$\begin{aligned}
&\left|\boldsymbol{b}_n^{2,l+1,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\cdots,\boldsymbol{b}_n^{2,l+1,m+n-1},\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{e}_n\right|\\
&\quad=(-1)^n\left|\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{b}_n^{2,l+1,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\cdots,\boldsymbol{b}_n^{2,l+1,m+n-1},\boldsymbol{e}_n\right| &(\mathrm{A.44})\\
&\quad=\frac{(-1)^n}{{c_1}^2aq^{-l+m+3}}\left|\boldsymbol{b}_{n-1}^{1,l,m+1},\boldsymbol{b}_{n-1}^{1,l,m+2},\boldsymbol{b}_{n-1}^{2,l+1,m+3},\boldsymbol{b}_{n-1}^{2,l+1,m+4},\cdots,\boldsymbol{b}_{n-1}^{2,l+1,m+n-1}\right|. &(\mathrm{A.45})
\end{aligned}$$

Repeating this procedure, we obtain (A.41).

We next show (A.42). From (A.40), we have

$$\begin{aligned}
&\left|\boldsymbol{b}_n^{2,l+1,m+1},\boldsymbol{b}_n^{2,l+1,m+2},\cdots,\boldsymbol{b}_n^{2,l+1,m+n-1},\boldsymbol{b}_n^{1,l,m+1}\right|\\
&\quad=(-1)^n\left|\boldsymbol{b}_n^{2,l+1,m+1},\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{b}_n^{2,l+1,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\cdots,\boldsymbol{b}_n^{2,l+1,m+n-1}\right| &\text{(A.46)}\\
&\quad=(-1)^{n+1}\frac{1+{c_1}^2a^2q^{2m+2}}{{c_1}^2aq^{-l+m+2}}\left|\boldsymbol{b}_n^{1,l,m},\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{b}_n^{2,l+1,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\cdots,\boldsymbol{b}_n^{2,l+1,m+n-1}\right| &\text{(A.47)}\\
&\quad=(-1)^{n+1}\frac{1+{c_1}^2a^2q^{2m+2}}{{c_1}^2aq^{-l+m+2}}\,\frac{1}{{c_1}^2aq^{-l+m+3}}\\
&\qquad\times\left|\boldsymbol{b}_n^{1,l,m},\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{b}_n^{1,l,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\boldsymbol{b}_n^{2,l+1,m+4},\cdots,\boldsymbol{b}_n^{2,l+1,m+n-1}\right|. &\text{(A.48)}
\end{aligned}$$

Repeating this procedure, we obtain (A.42).

Finally (A.43) can be obtained in a similar manner by using (A.40) as

$$\begin{aligned}
&\left|\boldsymbol{b}_n^{2,l+1,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\cdots,\boldsymbol{b}_n^{2,l+1,m+n},\boldsymbol{b}_n^{1,l,m+1}\right|\\
&\quad=(-1)^{n-1}\left|\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{b}_n^{2,l+1,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\cdots,\boldsymbol{b}_n^{2,l+1,m+n}\right| &\text{(A.49)}\\
&\quad=\frac{(-1)^{n-1}}{{c_1}^2aq^{-l+m+3}}\left|\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{b}_n^{1,l,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\boldsymbol{b}_n^{2,l+1,m+4},\cdots,\boldsymbol{b}_n^{2,l+1,m+n}\right| &\text{(A.50)}\\
&\quad=\frac{(-1)^{n-1}}{{c_1}^2aq^{-l+m+3}}\,\frac{1}{{c_1}^2aq^{-l+m+4}}\\
&\qquad\times\left|\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{b}_n^{1,l,m+2},\boldsymbol{b}_n^{1,l,m+3},\boldsymbol{b}_n^{2,l+1,m+4},\boldsymbol{b}_n^{2,l+1,m+5},\cdots,\boldsymbol{b}_n^{2,l+1,m+n}\right|, &\text{(A.51)}
\end{aligned}$$

which completes the proof. ■

Now we express $\psi_n^{l,m}$ by $b_n^{1,l,m}$. From (A.40), we obtain

$$\begin{aligned}
\psi_n^{l,m}&=\begin{vmatrix}
I_{l,m} & q^{-1}I_{l+1,m+1} & \cdots & q^{-(n-1)^2}I_{l+n-1,m+n-1}\\
I_{l+1,m} & q^{-2}I_{l+2,m+1} & \cdots & q^{-n(n-1)}I_{l+n,m+n-1}\\
\vdots & \vdots & \ddots & \vdots\\
I_{l+n-1,m} & q^{-n}I_{l+n,m+1} & \cdots & q^{-2(n-1)^2}I_{l+2n-2,m+n-1}
\end{vmatrix} &\text{(A.52)}\\
&=\left|\boldsymbol{b}_n^{1,l,m},\boldsymbol{b}_n^{2,l+1,m+1},\cdots,\boldsymbol{b}_n^{n,l+n-1,m+n-1}\right| &\text{(A.53)}\\
&=\left(\prod_{k=1}^{n-1}\frac{1}{({c_1}^2aq^{-l+m+2k})}\right)\left|\boldsymbol{b}_n^{1,l,m},\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{b}_n^{2,l+1,m+2},\cdots,\boldsymbol{b}_n^{n-1,l+n-2,m+n-1}\right|. &\text{(A.54)}
\end{aligned}$$

Repeating this procedure, we obtain

$$\left|\boldsymbol{b}_n^{1,l,m},\boldsymbol{b}_n^{1,l,m+1},\cdots,\boldsymbol{b}_n^{1,l,m+n-1}\right|=\left(\prod_{r=1}^{n-1}\prod_{k=1}^{r}({c_1}^2aq^{-l+m+2k+n-r-1})\right)\psi_n^{l,m}. \tag{A.55}$$

Now it is possible to derive the bilinear equations for $\psi_n^{l,m}$ as follows:

Lemma A.5. *The following bilinear equations hold*:

$$\psi_{n+1}^{l,m}\psi_n^{l+1,m+1}-\frac{\psi_{n+1}^{l+1,m+1}\psi_n^{l,m}}{aq^{m+n}(1-q^{l+n+1})}+\frac{\psi_{n+1}^{l+1,m}\psi_n^{l,m+1}}{aq^{m+n}(1-q^{l+n+1})}=0, \tag{A.56}$$

$$\psi_{n+1}^{l,m}\psi_n^{l+1,m+2}+\frac{{c_1}^2aq^{-l+m-n+1}}{1+{c_1}^2a^2q^{2m+2}}\,\psi_{n+1}^{l+1,m+1}\psi_n^{l,m+1}-\frac{\psi_{n+1}^{l,m+1}\psi_n^{l+1,m+1}}{1+{c_1}^2a^2q^{2m+2}}=0. \tag{A.57}$$

Proof. We first prove (A.56). From the Plücker relation, we have

$$\begin{aligned}
&\left|\boldsymbol{a}_n^{l,m},\boldsymbol{a}_n^{l,m+1},\cdots,\boldsymbol{a}_n^{l,m+n-1}\right|\left|\boldsymbol{a}_n^{l,m+1},\boldsymbol{a}_n^{l,m+2},\cdots,\boldsymbol{a}_n^{l,m+n-2},\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{e}_n\right|\\
&\quad-\left|\boldsymbol{a}_n^{l,m},\boldsymbol{a}_n^{l,m+1},\cdots,\boldsymbol{a}_n^{l,m+n-2},\boldsymbol{b}_n^{1,l+1,m+1}\right|\left|\boldsymbol{a}_n^{l,m+1},\boldsymbol{a}_n^{l,m+2},\cdots,\boldsymbol{a}_n^{l,m+n-1},\boldsymbol{e}_n\right|\\
&\quad+\left|\boldsymbol{a}_n^{l,m},\boldsymbol{a}_n^{l,m+1},\cdots,\boldsymbol{a}_n^{l,m+n-2},\boldsymbol{e}_n\right|\left|\boldsymbol{a}_n^{l,m+1},\boldsymbol{a}_n^{l,m+2},\cdots,\boldsymbol{a}_n^{l,m+n-1},\boldsymbol{b}_n^{1,l+1,m+1}\right|=0,
\end{aligned}\tag{A.58}$$

and then from (A.23), (A.24), (A.28), (A.29) and (A.30), we get

$$\begin{aligned}
&aq^{m+n-1}(1-q^{l+n})\left|\boldsymbol{b}_n^{1,l,m},\boldsymbol{b}_n^{1,l,m+1},\cdots,\boldsymbol{b}_n^{1,l,m+n-1}\right|\left|\boldsymbol{b}_{n-1}^{1,l+1,m+1},\boldsymbol{b}_{n-1}^{1,l+1,m+2},\cdots,\boldsymbol{b}_{n-1}^{1,l+1,m+n-1}\right|\\
&\quad+q^{n-1}\left|\boldsymbol{b}_n^{1,l+1,m},\boldsymbol{b}_n^{1,l+1,m+1},\cdots,\boldsymbol{b}_n^{1,l+1,m+n-1}\right|\left|\boldsymbol{b}_{n-1}^{1,l,m+1},\boldsymbol{b}_{n-1}^{1,l,m+2},\cdots,\boldsymbol{b}_{n-1}^{1,l,m+n-1}\right|\\
&\quad-\left|\boldsymbol{b}_{n-1}^{1,l,m},\boldsymbol{b}_{n-1}^{1,l,m+1},\cdots,\boldsymbol{b}_{n-1}^{1,l,m+n-2}\right|\left|\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{b}_n^{1,l+1,m+2},\cdots,\boldsymbol{b}_n^{1,l+1,m+n}\right|=0,
\end{aligned}\tag{A.59}$$

which yields (A.56) by using (A.55).

We next prove (A.57). We have from the Plücker relation

$$\begin{aligned}
&\left|\boldsymbol{b}_n^{2,l+1,m+1},\boldsymbol{b}_n^{2,l+1,m+2},\cdots,\boldsymbol{b}_n^{2,l+1,m+n}\right|\left|\boldsymbol{b}_n^{2,l+1,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\cdots,\boldsymbol{b}_n^{2,l+1,m+n-1},\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{e}_n\right|\\
&\quad-\left|\boldsymbol{b}_n^{2,l+1,m+1},\boldsymbol{b}_n^{2,l+1,m+2},\cdots,\boldsymbol{b}_n^{2,l+1,m+n-1},\boldsymbol{b}_n^{1,l,m+1}\right|\left|\boldsymbol{b}_n^{2,l+1,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\cdots,\boldsymbol{b}_n^{2,l+1,m+n},\boldsymbol{e}_n\right|\\
&\quad+\left|\boldsymbol{b}_n^{2,l+1,m+1},\boldsymbol{b}_n^{2,l+1,m+2},\cdots,\boldsymbol{b}_n^{2,l+1,m+n-1},\boldsymbol{e}_n\right|\left|\boldsymbol{b}_n^{2,l+1,m+2},\boldsymbol{b}_n^{2,l+1,m+3},\cdots,\boldsymbol{b}_n^{2,l+1,m+n},\boldsymbol{b}_n^{1,l,m+1}\right|=0.
\end{aligned}\tag{A.60}$$

By using (A.25), (A.26), (A.41), (A.42) and (A.43), we get

$$\begin{aligned}
&\left|\boldsymbol{b}_n^{1,l+1,m+1},\boldsymbol{b}_n^{1,l+1,m+2},\cdots,\boldsymbol{b}_n^{1,l+1,m+n}\right|\left|\boldsymbol{b}_{n-1}^{1,l,m+1},\boldsymbol{b}_{n-1}^{1,l,m+2},\cdots,\boldsymbol{b}_{n-1}^{1,l,m+n-1}\right|\\
&\quad+\frac{1+{c_1}^2a^2q^{2m+2}}{{c_1}^2aq^{-l+m-n+2}}\left|\boldsymbol{b}_n^{1,l,m},\boldsymbol{b}_n^{1,l,m+1},\cdots,\boldsymbol{b}_n^{1,l,m+n-1}\right|\left|\boldsymbol{b}_{n-1}^{1,l+1,m+2},\boldsymbol{b}_{n-1}^{1,l+1,m+3},\cdots,\boldsymbol{b}_{n-1}^{1,l+1,m+n}\right|\\
&\quad-\frac{1}{{c_1}^2aq^{-l+m+1}}\left|\boldsymbol{b}_{n-1}^{1,l+1,m+1},\boldsymbol{b}_{n-1}^{1,l+1,m+2},\cdots,\boldsymbol{b}_{n-1}^{1,l+1,m+n-1}\right|\left|\boldsymbol{b}_n^{1,l,m+1},\boldsymbol{b}_n^{1,l,m+2},\cdots,\boldsymbol{b}_n^{1,l,m+n}\right|=0.
\end{aligned}\tag{A.61}$$

Equation (A.57) is derived from (A.61) by using (A.55). This completes the proof. ■

By using (A.56) and (A.57), we prove Lemma 1.1. We note here that from the definition we have

$$\psi_n^{0,m}=\psi_n^{0,0}.\tag{A.62}$$

Substituting $l=0$ in (A.56) yields

$$\psi_{n+1}^{1,m+1}\psi_n^{0,0}=aq^{m+n}(1-q^{n+1})\psi_{n+1}^{0,0}\psi_n^{1,m+1}+\psi_{n+1}^{1,m}\psi_n^{0,0}.\tag{A.63}$$

Further, putting $l=0$ in (A.57), we have

$$\psi_n^{1,m+2}=\frac{\psi_n^{1,m+1}}{1+{c_1}^2a^2q^{2m+2}}-\frac{{c_1}^2aq^{m-n+1}}{1+{c_1}^2a^2q^{2m+2}}\,\frac{\psi_{n+1}^{1,m+1}\psi_n^{0,0}}{\psi_{n+1}^{0,0}}\,,\tag{A.64}$$

or equivalently,

$$\psi_{n+1}^{1,m}\psi_n^{0,0} = -\frac{1+{c_1}^2a^2q^{2m}}{{c_1}^2aq^{m-n}}\,\psi_{n+1}^{0,0}\psi_n^{1,m+1} + \frac{1}{{c_1}^2aq^{m-n}}\,\psi_{n+1}^{0,0}\psi_n^{1,m}. \tag{A.65}$$

Eliminating $\psi_{n+1}^{1,m+1}\psi_n^{0,0}$ from (A.63) and (A.64), we obtain

$$\psi_n^{1,m+2} = \frac{1-{c_1}^2(1-q^{n+1})a^2q^{2m+1}}{1+{c_1}^2a^2q^{2m+2}}\,\psi_n^{1,m+1} - \frac{{c_1}^2aq^{m-n+1}}{1+{c_1}^2a^2q^{2m+2}}\,\frac{\psi_{n+1}^{1,m}\ \psi_n^{0,0}}{\psi_{n+1}^{0,0}}\,, \tag{A.66}$$

and then by eliminating $\psi_{n+1}^{1,m}\psi_n^{0,0}$ from (A.65) and (A.66), the following linear relation for $\psi_n^{l,m}$ holds:

$$\psi_n^{1,m+2} = \frac{1+q+{c_1}^2a^2q^{2m+n+2}}{1+{c_1}^2a^2q^{2m+2}}\,\psi_n^{1,m+1} - \frac{q}{1+{c_1}^2a^2q^{2m+2}}\,\psi_n^{1,m}. \tag{A.67}$$

We rewrite (A.65) as

$$\frac{\psi_{n+1}^{0,0}}{\psi_n^{0,0}} = -\frac{{c_1}^2aq^{m-n+1}\psi_{n+1}^{1,m+1}}{(1+{c_1}^2a^2q^{2m+2})\psi_n^{1,m+2} - \psi_n^{1,m+1}}\,. \tag{A.68}$$

Eliminating $\psi_n^{1,m+2}$ from (A.67) and (A.68), we obtain

$$\frac{\psi_{n+2}^{0,0}}{\psi_{n+1}^{0,0}} = -\frac{{c_1}^2aq^{m-n-1}\psi_{n+2}^{1,m+1}}{(1+{c_1}^2a^2q^{2m+n+2})\psi_{n+1}^{1,m+1} - \psi_{n+1}^{1,m}}\,, \tag{A.69}$$

and then eliminating $\psi_{n+1}^{1,m}$ from (A.63) and (A.69), we obtain the following relation:

$$\frac{\psi_{n+2}^{0,0}\psi_{n+1}^{1,m+1}}{\psi_{n+1}^{0,0}\psi_{n+2}^{1,m+1}} = -\frac{{c_1}^2q^{-2n-1}}{{c_1}^2aq^{m+2} + (1-q^{n+1})\dfrac{\psi_{n+1}^{0,0}\psi_n^{1,m+1}}{\psi_n^{0,0}\psi_{n+1}^{1,m+1}}}\,. \tag{A.70}$$

Finally, replacing m by $-m-1$, we obtain

$$\frac{\psi_{n+2}^{0,0}\psi_{n+1}^{1,-m}}{\psi_{n+1}^{0,0}\psi_{n+2}^{1,-m}} = -\frac{{c_1}^2q^{-2n-1}}{{c_1}^2aq^{-m+1} + (1-q^{n+1})\dfrac{\psi_{n+1}^{0,0}\psi_n^{1,-m}}{\psi_n^{0,0}\psi_{n+1}^{1,-m}}}\,. \tag{A.71}$$

Putting

$$X_n = i\,\frac{(1-q^{n+1})q^n}{c_1}\,\frac{\psi_{n+1}^{0,0}\psi_n^{1,-m}}{\psi_n^{0,0}\psi_{n+1}^{1,-m}}\,, \tag{A.72}$$

(A.71) is rewritten as the following discrete Riccati equation:

$$X_{n+1} = \frac{1-q^{n+2}}{X_n + ic_1aq^{n-m+1}}\,. \tag{A.73}$$

We note here that in the case of $a_0a_1 = q^{-2N}$, q-P$_{\rm IV}$ (1.7) admits a specialization to the following discrete Riccati equation:

$$X_{n+1} = \frac{1-q^{-2N+n}a_2}{X_n + q^{-2N+n-m}{a_0}^{-1/2}a_2}\,. \tag{A.74}$$

This completes the proof of Lemma 1.1.

References

[1] D. Arinkin and A. Borodin, τ-function of discrete isomonodromy transformations and probability, Compos. Math. **145** (2009) 747–772.

[2] A. Borodin, Discrete gap probabilities and discrete Painlevé equations, Duke Math. J. **117** (2003) 489–542.

[3] A. Borodin and D. Boyarchenko, Distribution of the first particle in discrete orthogonal polynomial ensembles, Comm. Math. Phys. **234** (2003) 287–338.

[4] E. Brézin and V.A. Kazakov, Exactly solvable theories of closed strings, Phys. Lett. B **236** (1990) 144–150.

[5] Y. Chen and D. Dai, Painlevé V and a Pollaczek-Jacobi type orthogonal polynomials, J. Approx. Theory **162** (2010) 2149–2167.

[6] D. Dai and L. Zhang, Painlevé VI and Hankel determinants for the generalized Jacobi weight, J. Phys. A **43** (2010).

[7] G. Gasper and M. Rahman, Basic Hypergeometric Series (Second edition), Encyclopedia of Mathematics and Its Applications **96** (Cambridge University Press, Cambridge, 2004).

[8] B. Grammaticos, Y. Kosmann-Schwarzbach and T. Tamizhmani, Discrete integrable systems, Proceedings of the International School held in Pondicherry, February 2–14, 2003. Lecture Notes in Physics, **644**. Springer-Verlag, Berlin, 2004.

[9] B. Grammaticos, F.W. Nijhoff, V. Papageorgiou, A. Ramani and J. Satsuma, Linearization and solutions of the discrete Painlevé III equation, Phys. Lett. A **185** (1994) 446–452.

[10] B. Grammaticos, Y. Ohta, A. Ramani and H. Sakai, Degeneration through coalescence of the q-Painlevé VI equation, J. Phys. A **31** (1998) 3545–3558.

[11] B. Grammaticos, A. Ramani and V. Papageorgiou, Do integrable mappings have the Painlevé property?, Phys. Rev. Lett. **67** (1991) 1825–1828.

[12] C. Itzykson and J.B. Zuber, The planar approximation. II, J. Math. Phys. **21** (1980) 411–421.

[13] M. Jimbo, T. Miwa and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function, *Phys. D* **2** (1981) 306–352.

[14] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II, *Phys. D* **2** (1981) 407–448.

[15] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III, *Phys. D* **4** (1981/82) 26–46.

[16] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada, Hypergeometric solutions to the q-Painlevé equations, Int. Math. Res. Not. **2004** (2004) 2497–2521.

[17] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada, Construction of hypergeometric solutions to the q-Painlevé equations, Int. Math. Res. Not. **2005** (2005) 1441–1463.

[18] K. Kajiwara, M. Noumi and Y. Yamada, A study on the fourth q-Painlevé equation, *J. Phys. A: Math. Gen.* **34** (2001) 8563–8581.

[19] R. Koekoek, P.A. Lesky and R.F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.

[20] T. Masuda, Hypergeometric τ-functions of the q-Painlevé system of type $E_7^{(1)}$, *SIGMA Symmetry Integrability Geom. Methods Appl.* **5** (2009).

[21] T. Masuda, Hypergeometric τ-functions of the q-Painlevé system of type $E_8^{(1)}$, Ramanujan J. **24** (2011).

[22] T. Miwa, M. Jimbo and E. Date, Solitons (Cambridge University Press, Cambridge, 2000).

[23] N. Nakazono, Hypergeometric τ Functions of the q-Painlevé Systems of Type $(A_2{+}A_1)^{(1)}$, *SIGMA Symmetry Integrability Geom. Methods Appl.* **6** (2010).

[24] N. Nakazono, In preparation.

[25] N. Nakazono and S. Nishioka, Solutions to a q-analog of the Painlevé III equation of type $D_7^{(1)}$, Funkcial. Ekvac. (in press).

[26] F.W. Nijhoff and V.G. Papageorgiou, Similarity reductions of integrable lattices and discrete analogues of the Painlevé II equation, Phys. Lett. A **153** (1991) 337–344.

[27] M. Noumi, Painlevé equations through symmetry (American Mathematical Society, Providence, 2004).

[28] M. Noumi and Y. Yamada, Symmetries in the fourth Painlevé equation and Okamoto polynomials, *Nagoya Math. J.* **153** (1999) 53–86.

[29] Y. Ohyama, H. Kawamuko, H. Sakai, and K. Okamoto, Studies on the Painlevé equations. V. Third Painlevé equations of special type $P_{\rm III}(D_7)$ and $P_{\rm III}(D_8)$, J. Math. Sci. Univ. Tokyo **13** (2006) 145–204.

[30] K. Okamoto, Studies on the Painlevé equations. III. Second and Fourth Painlevé equation, $P_{\rm II}$ and $P_{\rm IV}$, Math. Ann. **275** (1986) 221–255.

[31] K. Okamoto, Studies on the Painlevé equations. I. Sixth Painlevé equation $P_{\rm VI}$, Ann. Mat. Pura Appl. **146** (1987) 337–381.

[32] K. Okamoto, Studies on the Painlevé equations. II. Fifth Painlevé equation $P_{\rm V}$, Japan. J. Math. (N.S.) **13** (1987) 47–76.

[33] K. Okamoto, Studies on the Painlevé equations. IV. Third Painlevé equation $P_{\rm III}$, Funkcial. Ekvac. **30** (1987) 305–332.

[34] V. Periwal and D. Shevitz, Unitary-matrix models as exactly solvable string theories, Phys. Rev. Lett. **64** (1990) 1326–1329.

[35] A. Ramani and B. Grammaticos, Discrete Painlevé equations: coalescences, limits and degeneracies, Physica A **228** (1996) 160–171.

[36] A. Ramani, B. Grammaticos and J. Hietarinta, Discrete versions of the Painlevé equations, Phys. Rev. Lett. **67** (1991) 1829–1832.

[37] H. Sakai, Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Comm. Math. Phys. **220** (2001) 165–229.

[38] J. Shohat, A differential equation for orthogonal polynomials, Duke Math. J. **5** (1939) 401–417.

[39] K.M. Tamizhmani, B. Grammaticos, A.S. Carstea and A. Ramani, The q-discrete Painlevé IV equations and their properties, Regul. Chaotic Dyn. **9** (2004) 13–20.

[40] T. Tsuda, Tau functions of q-Painlevé III and IV equations, Lett. Math. Phys. **75** (2006) 39–47.